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Information-Statistical Extended Thermodynamics and Turbulent-Induced Heat Flux Inhibition

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Abstract. It is shown that information theory predicts a decrease of the effective thermal conductivity. This new correction corresponds to a time-delayed vorticity dynamics, and it may explain the inhibition of the effective heat flux in turbulent systems.

1. Introduction

Turbulent systems are typically far from equilibrium, but nonequilibrium statistical theories are complicated and not completely well-established. This is why much work has been devoted to two-dimensional turbulence in the absence of viscosity: then, there are two constants of motion (namely the energy and the entropy) and one may derive approximate results by means of the methods of equilibrium statistical mechanics [1]. Nevertheless, this is not possible for the case of three-dimensional turbulence and, more specifically, if the role of viscous forces cannot be neglected. It means that topics such as the inhibition of the heat flux in three-dimensional turbulent flows [2] require a different perspective. One possibility is information theory, a statistical-mechanical method the validity of which is not restricted to equilibrium systems [3]. It is true, however, that turbulence is an extremely broad topic encompassing many different issues such as turbulent diffusion, magnetohydrodynamic turbulence, etc., which will certainly complicate any framework aiming at complete generality. In fact, the same is true when dealing with radiative transfer: nonequilibrium radiation is of utmost importance in the description of many systems, such as stellar atmospheres, shock-wave thermometry, etc. However, information theory has been applied in order to obtain a thermodynamically-consistent description of radiative transfer from which one can derive specific results, at least in simple situations [4, 5]. We will here present an analogous approach to turbulence, keeping in mind that we are looking for a reasonable, simple formalism that allows us to focus our attention on the fundamental points. Our analysis is also motivated by the fact that recently, simple thermodynamic approaches have made it possible to analyze several properties and effects of turbulent flows [6, 7].

2. Information-Statistical Approach to Turbulence

An important feature of turbulent flows is that small, uncontrollable disturbances in the initial conditions lead to different values for hydrodynamic variables, e.g. for the velocity $\boldsymbol{\vartheta}(\boldsymbol{x}, t)$ of the fluid at position $\boldsymbol{x}$ and time t. This justifies a statistical approach to turbulence, within which one introduces an ensemble of values of, say, $\boldsymbol{\vartheta}(\boldsymbol{x}, t)$, obtained in the same experiment and under the same conditions [8]. In addition to $\boldsymbol{\vartheta}(\boldsymbol{x}, t)$, let us introduce the density and specific internal energy fields as $\widehat{\rho}(\boldsymbol{x}, t)$ and $\widehat{u}(\boldsymbol{x}, t)$, respectively. We define Γ as a five-vector with components $\widehat{\rho}(\boldsymbol{x}, t)$, $\widehat{u}(\boldsymbol{x}, t)$ and the components of $\boldsymbol{\vartheta}(\boldsymbol{x}, t)$, so that $d\Gamma \equiv d^3\vartheta \, d\widehat{\rho} \, d\widehat{u}$. Let $f(\Gamma)\, d\Gamma$ stand for the (time- and space-dependent) probability that, for a given experiment and initial conditions, the velocity, specific internal energy and density of the fluid have values in a differential $d\Gamma$, centered at Γ.

We require the probability to be normalized,

$$1 = \int d\Gamma \, f(\Gamma) \, . \tag{1}$$

The expected value for the kinetic energy per unit mass is

$$k(\boldsymbol{x}, t) = \frac{v^2(\boldsymbol{x}, t)}{2} = \int d\Gamma \, f(\Gamma) \, \frac{|\boldsymbol{\vartheta}(\boldsymbol{x}, t)|^2}{2} \, , \tag{2}$$

where $v^2(\boldsymbol{x}, t)$ is the expected value for $|\boldsymbol{\vartheta}(\boldsymbol{x}, t)|^2$. The specific kinetic energy (2) of the fluid has been used by many authors in statistical approaches to turbulence. Sometimes the temperature is introduced as the partial derivative of the entropy with respect to the kinetic energy (2) (see, e.g., [1] and references therein). However, it is well-known from both the classical [9] and the extended [10] theories of nonequilibrium thermodynamics that the temperature is not related to the kinetic energy (2) but to the internal energy. It is true that this is not the most general possible situation as it is clear from the fact that, in the case of radiation-matter systems, both the matter internal energy and the radiation energy have to be taken into account [5]. However, turbulence takes place in fluids, thus it seems reasonable to require that any approach to turbulence should be in agreement with well-established, general results for fluids. We therefore see that, from a thermodynamic perspective, the expected value for the specific internal energy, namely

$$u = \int d\Gamma\, f(\Gamma)\, \widehat{u}(\Gamma) \tag{3}$$

(instead of (2)), should be necessary if the temperature is to be introduced in a way consistent with the usual one in non-turbulent fluid theory.

An important quantity in turbulent flows is the vorticity field. We write the expected value for the vorticity of the flow as [1]

$$\nabla \times \boldsymbol{v}(\boldsymbol{x}, t) = \int d\Gamma\, f(\Gamma) \nabla \times \boldsymbol{\vartheta}\,. \tag{4}$$

Finally, the expected density field is

$$\rho(\boldsymbol{x}, t) = \int d\Gamma\, f(\Gamma) \widehat{\rho}\,. \tag{5}$$

According to information theory, the most probable distribution $f(\Gamma)$ is that which maximizes the functional [3]

$$s(\boldsymbol{x}, t) = -k \int d\Gamma\, f(\Gamma) \ln f(\Gamma) \tag{6}$$

under the constraints of the system. The expression (6) is of the form

$$s \propto -\sum_i p_i \ln p_i\,,$$

where p_i is the probability of microstate i, which is just the Boltzmann expression for the entropy [11]. Thus k in (6) is a positive constant that may be chosen arbitrarily and, with a suitable choice for k, s may be interpreted as the specific entropy of the system [12]. The entropy formula (6) has been used in the analysis of spectral energy transport in turbulent flows and leads, as a special case, to the Kolmogorov spectrum [12]. Here, we will not need spectral decomposition because we are interested in studying i) the thermodynamics of the system, and ii) the influence of the vorticity field on the transport properties of turbulent fluids. We stress that we make use of the constraint (4) because of the well-known relevance of the vorticity field in turbulent flows, and of the constraints (3) and (5) because the specific internal energy and the density are of central importance in any thermodynamic description of fluid systems. Of course, additional constraints (even an infinite number [13]) could be included, and they would lead to more general approaches. However, as stressed in the Introduction, we look for a specific and the simplest possible case in order to explore the consequences of information theory in the thermodynamic description of turbulence. The use of information theory amounts to finding out the most probable flow under the set of constraints under consideration. Maximization of the specific entropy (6) under the constraints (1) and (3)–(5) yields

$$f(\Gamma) = Z \exp\Big[-\beta \widehat{u} - \boldsymbol{\gamma} \cdot (\nabla \times \boldsymbol{\vartheta}) - \varepsilon \widehat{\rho} \Big]\,. \tag{7}$$

where $Z = \exp[-1 - \lambda]$ is a normalization factor, and λ, β, γ and ε are Lagrange multipliers. It is worth stressing that the information theory approach is of purely probabilistic nature [3]: the result (7) does not depend on the identification of s in (6) with the entropy. In the present paper, however, such an identification (which has been widely used for both matter and radiation systems, see e.g. [4]) will be useful in order to find out the multipliers β, γ and ε in terms of measurable quantities.

From (6) we obtain for the difference in s between two macrostates of the flow with similar properties (i.e., with similar probability distributions $f(\Gamma)$)

$$\delta s \; = \; -k \int d\Gamma \, [\ln f(\Gamma) + 1] \, \delta f(\Gamma) \, , \tag{8}$$

which can be written, using (7), (1), (3) and (4), as

$$\delta s \; = \; k\beta \, \delta u + k\boldsymbol{\gamma} \cdot \delta(\boldsymbol{\nabla} \times \boldsymbol{v}) + k\varepsilon \, \delta\rho \, . \tag{9}$$

The radiative analog of this expression is (11) in [5]. Similarly to what was done in [5], we can identify the Lagrange multiplier β appearing in this expression from the thermodynamic definition of temperature T, namely [10] $1/T = \partial s / \partial u$. This yields, making use of (9),

$$\beta \; = \; \frac{1}{kT} \, . \tag{10}$$

Similarly, from the thermodynamic definition of pressure p, i.e. $p/T \equiv \partial s / \partial (1/\rho)$, we obtain

$$\varepsilon \; = \; -\frac{p}{kT\rho^2} \, .$$

From (9) and (10) we obtain

$$ds \; = \; \frac{1}{T} du + \frac{p}{T} \, dv + k\boldsymbol{\gamma} \cdot d(\boldsymbol{\nabla} \times \boldsymbol{v}) \, , \tag{11}$$

where $v = 1/\rho$ is the specific volume, and we have made use of, e.g., the symbol ds instead of δs because, in contrast to what happened in (8), the phase-space differential $d\Gamma$ no longer appears so that no confusion can arise between differentials referring to the phase space volume and those referring to macroscopic quantities. Equation (11) has the form which is well-known from extended irreversible thermodynamics [10]: it is nothing but the usual Gibbs equation from classical (or local-equilibrium) irreversible thermodynamics and an additional term (the last term in (11)) which is well-known in many situations, such as non-Fickian diffusion, fast heat or electrical conduction, etc.; such a term contains the differential of a variable which is relevant in the process under consideration: whereas an equation such as (11) is well-known with the last term containing, e.g., the differential of the electric current or that of the diffusion flux [10], here it has been derived and will be applied for the first time to the analysis of the role of vorticity in turbulent flows. It is important to note that, according to the result (11), the nonequilibrium specific entropy s is a function of the vorticity in addition to the specific internal energy and specific volume of the flow, i.e.

$$s = s(u, v, \nabla \times \boldsymbol{v}) , \tag{12}$$

and, also from (11), the Lagrange multiplier γ can be written as

$$\gamma = \frac{1}{k} \frac{\partial s(u, v, \nabla \times \boldsymbol{v})}{\partial(\nabla \times \boldsymbol{v})} . \tag{13}$$

We may compare the dependence (12) with the local-equilibrium assumption, which is the starting point of classical irreversible thermodynamics. According to this assumption, in states close enough to equilibrium the specific entropy depends locally on the same variables as those upon which it depends in equilibrium, i.e. [9]

$$s_{\mathrm{le}} = s_{\mathrm{le}}(u, v) , \tag{14}$$

where the subindex le denotes "local equilibrium". In the special case $\nabla \times \boldsymbol{v} = 0$, the generalized entropy (12) must reduce to its classical value (14), which does not depend on the additional variable $\nabla \times \boldsymbol{v}$. Thus, according to (13),

$$\gamma \Big|_{\nabla \times \boldsymbol{v} = 0} = 0 .$$

Therefore, we may consider a Taylor expansion for γ of the form

$$\gamma = \psi(\nabla \times \boldsymbol{v}) + O(\nabla \times \boldsymbol{v})^2 , \tag{15}$$

where $\psi = \psi(u, v)$ is a scalar field and $O(\nabla \times \boldsymbol{v})^2$ stands for second- and higher order terms. From (11) and (15) we obtain for the rate of increase of specific entropy

$$\frac{ds}{dt} = \frac{1}{T}\frac{du}{dt} + \frac{p}{T}\frac{d}{dt}\Big(\frac{1}{\rho}\Big) + k\psi(\nabla \times \boldsymbol{v}) \cdot \frac{d}{dt}(\nabla \times \boldsymbol{v}) . \tag{16}$$

On the other hand, the law of balance of mass is

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \boldsymbol{v} , \tag{17}$$

and that of the internal energy reads (see, e.g., [9], Chapter XII)

$$\rho \frac{du}{dt} = -\nabla \cdot \boldsymbol{q} - p \nabla \cdot \boldsymbol{v} - p^{\nu} \nabla \cdot \boldsymbol{v} - \overset{0}{\mathrm{P}}{}^{s\nu} : \overset{0}{\mathrm{V}} - \mathbf{P}^{a\nu} \cdot (\nabla \times \boldsymbol{v} - 2\boldsymbol{\omega}) , \tag{18}$$

where $\boldsymbol{q}$ is the heat flux, p is the pressure and p^{ν} is the scalar viscous pressure. $\overset{0}{\mathrm{P}}{}^{s\nu}$ is the symmetric (traceless) part of the viscous pressure tensor, and $\mathbf{P}^{a\nu}$ is the axial vector corresponding to its antisymmetric part [9]. $\boldsymbol{\omega}$ is the local angular velocity, and its contribution to the last term in (18) has been shown to be relevant for polyatomic fluids in which the internal structure plays an important role ($\boldsymbol{\omega}$ is then identified with the spin, which arises from internal molecular rotations, and it leads to an internal contribution to the angular momentum per unit mass of the fluid) [14, 15]. Since we are here interested in the consequences of the vorticity $\boldsymbol{\nabla}\times\boldsymbol{v}$ on the properties of turbulent fluids, we will for simplicity consider situations such that the contribution of $\boldsymbol{\omega}$ is negligible, i.e.,

$$\rho\frac{du}{dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{q} - p\boldsymbol{\nabla}\cdot\boldsymbol{v} - p^{\nu}\boldsymbol{\nabla}\cdot\boldsymbol{v} - \overset{0}{\mathrm{P}}{}^{s\nu} : \overset{0}{\mathrm{V}} - \mathbf{P}^{a\nu}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) . \tag{19}$$

Use of (17) and (19) into the information-statistical result (16) yields

$$\begin{aligned}\frac{ds}{dt} &= -\frac{\boldsymbol{\nabla}\cdot\boldsymbol{q}}{\rho T} - \frac{p^{\nu}}{\rho T}\boldsymbol{\nabla}\cdot\boldsymbol{v} - \frac{\overset{0}{\mathrm{P}}{}^{s\nu} : \overset{0}{\mathrm{V}}}{\rho T} - \left(\frac{\mathbf{P}^{a\nu}}{\rho T}\right)\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) \\ &\quad + \psi k(\boldsymbol{\nabla}\times\boldsymbol{v})\cdot\frac{d}{dt}(\boldsymbol{\nabla}\times\boldsymbol{v}) .\end{aligned} \tag{20}$$

Let $\boldsymbol{J}^{s}$ and σ^{s} stand for the local entropy flux and the local entropy rate of production, respectively. The general form of the entropy balance equation is

$$\rho\frac{ds}{dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}^{s} + \sigma^{s} .$$

Comparison with (20) leads to the identifications

$$\boldsymbol{J}^{s} = \frac{\boldsymbol{q}}{T}$$

and

$$\sigma^{s} = \boldsymbol{q}\cdot\boldsymbol{\nabla}\Big(\frac{1}{T}\Big) - \frac{p^{\nu}}{T}\boldsymbol{\nabla}\cdot\boldsymbol{v} - \frac{\overset{0}{\mathrm{P}}{}^{s\nu} : \overset{0}{\mathrm{V}}}{T} - \Big[\mathbf{P}^{a\nu} - k\rho T\psi\frac{d}{dt}(\boldsymbol{\nabla}\times\boldsymbol{v})\Big]\cdot\frac{(\boldsymbol{\nabla}\times\boldsymbol{v})}{T} . \tag{21}$$

The second law requires that $\sigma^{s}\geq 0$. From (21), we see that this will be fulfilled if the fluid follows the simple evolution equations

$$\boldsymbol{q} = -\lambda\boldsymbol{\nabla}T \qquad (\lambda\geq 0) , \tag{22}$$

$$p^{\nu} = -\xi\boldsymbol{\nabla}\cdot\boldsymbol{v} \qquad (\xi\geq 0) , \tag{23}$$

$$\overset{0}{\mathrm{P}}{}^{s\nu} = -2\eta\,\overset{0}{\mathrm{V}} \qquad (\eta\geq 0) , \tag{24}$$

$$\mathbf{P}^{a\nu} - k\rho T\psi\frac{d}{dt}(\boldsymbol{\nabla}\times\boldsymbol{v}) = -\mu(\boldsymbol{\nabla}\times\boldsymbol{v}) \qquad (\mu\geq 0) . \tag{25}$$

The first three equations are the usual Fourier, Stokes and Newton laws (we may thus identify λ, ξ and η with the thermal conductivity, bulk viscosity and shear viscosity, respectively). It is worth stressing that consistency with these well-known results would not have been reached if we had made use of the specific energy density, namely $v^2/2$ (see (2)), instead of the internal energy (3). This fact is not surprising in view of the remarks presented in the Introduction, and justifies our departure from previous work (see, e.g., [1] and references therein). On the other hand, (25) corresponds to the effect of the vorticity. In order to understand the physical meaning of this equation, it is useful to recall that, as stressed in the last section of the treatise on turbulence by Monin and Yaglom [8], transport equations of linear form between fluxes and forces correspond to an infinite spread of propagation of signals. Physically, however, one should expect fluxes to be delayed with respect to their thermodynamically conjugate forces. For example, a linear relationship between the axial vector $\mathbf{P}^{a\nu}$ and the vorticity (for the case $\omega \ll |\nabla \times \boldsymbol{v}|$, considered above) is well-known and usually written as [9]

$$(\nabla \times \boldsymbol{v}) = -\frac{1}{\eta_r}\mathbf{P}^{a\nu}, \tag{26}$$

where η_r is called the rotational viscosity. This law corresponds to the case in which a nonvanishing vorticity instantaneously gives rise to an associate stress. It is more reasonable, however, that this effect will take place after a time delay τ, i.e.

$$(\nabla \times \boldsymbol{v})_i(\boldsymbol{x} - \boldsymbol{v}\tau,\, t - \tau) = -\frac{1}{\eta_r}P_i^{a\nu}(\boldsymbol{x}, t),$$

where $\boldsymbol{x}$ is the position vector, t is the time and $i = x, y, z$. After Taylor-expanding the left-hand side and retaining first-order terms we obtain

$$(\nabla \times \boldsymbol{v})_i - \Big[\nabla(\nabla \times \boldsymbol{v})_i \cdot \boldsymbol{v} + \frac{\partial}{\partial t}(\nabla \times \boldsymbol{v})_i\Big]\tau = -\frac{1}{\eta_r}P_i^{a\nu},$$

where all quantities are evaluated at $(\boldsymbol{x}, t)$. This equation may also be written as

$$\mathbf{P}^{a\nu} - \eta_r\tau\frac{d}{dt}(\nabla \times \boldsymbol{v}) = -\eta_r(\nabla \times \boldsymbol{v}), \tag{27}$$

which is just (25) provided that we identify μ and ψ in terms of measurable quantities as

$$\mu = \eta_r, \tag{28}$$

$$\psi = \frac{\tau\eta_r}{k\rho T}, \tag{29}$$

so that the extended Gibbs equation (11) may be written, making use also of (15),

$$ds = \frac{1}{T}du + \frac{p}{T}dv + \frac{\tau\eta_r}{\rho T}(\nabla \times \boldsymbol{v}) \cdot d(\nabla \times \boldsymbol{v}). \tag{30}$$

The time-delayed equation (25) (or (27)) is analogous to the radiative equation (38) in [5]. Both results have been derived from information-statistical theory, although we would like to stress that time-delayed transport is well-known to yield reliable predictions for many phenomena, e.g. hyperbolic heat conduction [16, 17] and shear-wave speeds in liquids [18]. In fact, it would not have been difficult to obtain time-delayed more general results for other fluxes (instead of the linear laws (22)–(24)) simply by making use of additional variables in our formalism. However, the simple approach presented here will be enough in order to analyze the effect of the vorticity, which is discussed in the next section.

3. Application. Turbulent-Induced Heat Flux Inhibition

A basic result of turbulence theory is that the transport properties of turbulent flows are different from those of laminar ones: the Reynolds decomposition of fluid variables leads to additional terms in the transport equations for the mean variables, as compared to the corresponding equations for the full variables. Such additional terms, which arise from turbulent fluctuations, lead to the introduction of an eddy or turbulent heat conductivity, viscosity, etc., which usually correspond to an enhancement of, e.g., the heat flux in turbulent flows [8]. However, as stressed above, the finite speed of thermal signals implies that the heat flux cannot reach arbitrarily high values. Indeed, the energy density times the maximum molecular speed gives an upper, finite bound for the rate of energy transfer. It means that in general, there will be one or several mechanisms at work that make it impossible for the heat flux to acquire arbitrarily high values: heat-flux inhibitions are well-known in radiative transfer [19], plasma physics [20] and astrophysics [21]. The presence of turbulence has also been experimentally associated with a decrease in the effective thermal conductivity in plasmas [2]. While one may develop kinetic-theoretical approaches that carefully take into account all of the microscopic processes, it would be interesting to try to approach the topic of nonlinear conductivities arising from turbulence from a thermodynamic, mathematically simple perspective. Once the main point, namely a possible turbulent heat-inhibiting mechanism, has been approached (as done below), it should be possible to complicate our basic model by the introduction of additional variables (e.g., electric and magnetic fields, several particle species, and so on) in order to deal with specific systems (e.g., plasmas). Before considering the consequences of our model in turbulent flows, however, it is useful to resort to Eu's theory of nonlinear transport [20, 10]. This theory is consistent with extended irreversible thermodynamics and applies to a wide range of densities. Within its framework, the result for transport coefficients such as the thermal conductivity λ is [10]

$$\lambda = \lambda_0 \frac{\sqrt{\tau \sigma_\tau^s / n k_B}}{\sinh \sqrt{\tau \sigma_\tau^s / n k_B}}, \tag{31}$$

where λ_0 is the thermal conductivity in the absence of the time-delayed process in the system under consideration, n is the molecular number density, k_B is the Boltzmann constant, and σ_τ^s is the rate of entropy production for the relaxational or nonlinear process with time delay τ. In the simple case considered in the present paper, the vorticity dynamics is time-delayed (see (25) or (27)), and its corresponding entropy production is given by the last term in (21), which may also be written (using (25) and (28)) as

$$\sigma_\tau^s = \frac{\eta_r}{T}(\boldsymbol{\nabla} \times \boldsymbol{v})^2 .$$

We make use of this expression in (31). Since third- and higher order terms have been neglected (see (15) and (21)), we Taylor-expand the expression for λ up to second order. This yields

$$\lambda = \lambda_0\Big[1 - \frac{\tau\eta_r}{6nk_BT}(\boldsymbol{\nabla} \times \boldsymbol{v})^2\Big] + O(\boldsymbol{\nabla} \times \boldsymbol{v})^3 . \tag{32}$$

This equation predicts a reduction of the thermal conductivity λ due to the vorticity, and is analogous to the reduction of λ due to an electric current (as predicted by (12) in [22]). In the relevant case of turbulent flows, as mentioned above, a reduction of heat conduction has indeed been observed in the case of plasmas. Although we have stressed that, for the specific case of plasma physics, our model should be extended in order to include additional relevant variables (electric and magnetic fields, ion and electron concentrations, etc.), we can analyze the effect (32) within the simple model presented here. Use of (32) into (22) yields

$$\boldsymbol{q} \simeq \lambda_0\Big[1 - \frac{\lambda_v}{\lambda_0}\Big]\boldsymbol{\nabla} T , \tag{33}$$

where λ_v/λ_0 is the correction arising from the vorticity, i.e.

$$\lambda_v = \frac{\tau\eta_r}{6nk_BT}(\boldsymbol{\nabla} \times \boldsymbol{v})^2\lambda_0 . \tag{34}$$

Equation (33) is a generalization of the usual Fourier law of heat conduction. The term λ_v/λ_0 should be regarded as a small correction under the range of validity of the theory presented here, i.e. $\lambda_v/\lambda_0 \ll 1$ (otherwise, higher-order terms in (32) should be taken into account). This means that we can neglect, in a first approximation, additional nonlinear terms arising from this small correction when making use of (33) into the energy balance (19). In this way, we obtain a relatively simple equation,

$$\rho c\Big(\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} T\Big) = \lambda_0\Big[1 - \frac{\lambda_v}{\lambda_0}\Big]\nabla^2 T - \overset{0}{\mathrm{P}}{}^{s\nu} : \overset{0}{\mathrm{V}} - \mathbf{P}^{a\nu} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) , \tag{35}$$

where we have introduced the specific heat per unit mass c (i.e., $du = c\,dT$) and assumed an incompressible fluid (so that (17) implies $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$), also for the sake of mathematical simplicity.

In turbulent flows, one is interested in the mean properties of the fluid, which are easily measurable. The velocity field is decomposed into its mean $\boldsymbol{V} = \langle \boldsymbol{v} \rangle$ and fluctuating parts $\boldsymbol{v}'$, i.e. $\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{v}'$. Here $\langle \ldots \rangle$ denotes ensemble average. The other fields are similarly decomposed, e.g. $T = \overline{T} + T'$, $(\boldsymbol{\nabla} \times \boldsymbol{v}) = \overline{(\boldsymbol{\nabla} \times \boldsymbol{v})} + (\boldsymbol{\nabla} \times \boldsymbol{v})'$, etc. By making use of this decomposition and of (34) into (35), and following the usual procedure [8], we obtain the corresponding mean transport equation

$$\rho c \frac{d\overline{T}}{dt} = (\lambda_0 + \lambda_f - \lambda_{\overline{v}}) \nabla^2 \overline{T} - \overline{\overset{0}{\mathrm{P}}{}^{s\nu}} : \overline{\overset{0}{\mathrm{V}}} - \overline{\mathbf{P}^{a\nu}} \cdot \overline{(\boldsymbol{\nabla} \times \boldsymbol{v})}\,, \tag{36}$$

where, as usual [8], we have defined a contribution λ_f to the effective thermal conductivity, arising from the turbulent fluctuations, through

$$\lambda_f \nabla^2 \overline{T} = \rho c \boldsymbol{\nabla} \cdot \overline{(\boldsymbol{v}' T')} + \overline{\overset{0}{\mathrm{P}}{}^{s\nu\prime} : \overset{0}{\mathrm{V}}{}'} + \overline{\mathbf{P}^{a\nu\prime} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v})'} - \frac{\tau \eta_r \lambda_0}{6 n k_B} \overline{\left[(\boldsymbol{\nabla} \times \boldsymbol{v})^2 \right]' \left(\frac{\nabla^2 T}{T} \right)'}\,, \tag{37}$$

and we have analogously introduced an effective conductivity $\lambda_{\overline{v}}$ arising from the mean vorticity field through

$$\lambda_{\overline{v}} \nabla^2 \overline{T} = \frac{\tau \eta_r \lambda_0}{6 n k_B} \overline{(\boldsymbol{\nabla} \times \boldsymbol{v})^2}\, \overline{\left(\frac{\nabla^2 T}{T} \right)}\,. \tag{38}$$

The first term in the right-hand side of (37) is the usual Reynolds contribution [8], whereas the next two terms are due to the fluctuations of the viscous stresses. All of these terms would have been obtained also if one had simply assumed the usual (22)–(24) and (26). In contrast, the last term in (37) arises because of the vorticity fluctuations and is new from the model presented here, being ultimately due to the information-theoretical evolution equation (25). The relative importance of these terms will, of course, depend on the particular cases to which one wishes to apply the formalism in further work. It is important to stress that comparison of (36) and (35) shows that the effective thermal conductivity of the mean flow may be written as

$$\overline{\lambda} = \lambda_0 + \lambda_f - \lambda_{\overline{v}}\,,$$

so that in addition to the contribution λ_f arising from the turbulent fluctuations, our model predicts a modification in the effective conductivity, $\lambda_{\overline{v}}$, which shows that the effect of the vorticity may indeed lead to an inhibition of heat conduction in turbulent flows.

4. Conclusions

The analysis presented follows a spirit similar to that in previous papers [6, 7], in the sense that use has been made of a thermodynamic formalism. Our formalism has been derived from information theory and has been applied to the study the thermal conductivity of turbulent flows. The main conclusion of the paper is that information statistical theory leads to an extended Gibbs equation which corresponds to a decrease of the effective thermal conductivity. We have stressed that analogous results to those derived here can be found in many non-turbulent systems that may be properly described within the framework of extended irreversible thermodynamics. This analogy, including the information-theoretical foundations, is particularly clear in the case of radiative transfer [5]. Of course, more complicated systems require the use of additional variables, i.e. subsequent extensions of the simple model presented here, in the same way that particular radiative systems require more complicated theories that those presented in [4, 5]. Comparison of models, based on the presented formalism, and experimental data could be useful in order to explain heat-flux inhibitions in turbulent fluids and also as a possible way to measure the values of the vorticity relaxation time τ appearing in (27) and (34).

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