

Kinetic energy $mv^2/2$: resolution of a paradox

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Statement of the paradox

Consider a very simple energy problem: A mass m , which is initially at rest, falls down due to the Earth's gravity ($g = 9.8 \text{ m/s}^2$), and friction is negligible. Conservation of energy reads:

$$K_0 + P_0 = K + P, \quad (1)$$

where K_0 and P_0 are the kinetic and potential energies, respectively, at the initial position (height h), whereas K and P are the corresponding ones when the mass reaches the ground (height zero). Equation (1) can be written

$$mgh = \frac{mv^2}{2}, \quad (2)$$

which yields the well-known result for the final speed $v = \sqrt{2gh}$.

Now consider an observer who is moving with constant speed V relative to the ground (e.g., downwards). At first sight, we would argue that conservation of energy (as all physical laws) must hold in whatever frame, and that Equation (1) can be written in this observer's frame as

$$\frac{m(-V)^2}{2} + mgh = \frac{m(v-V)^2}{2}, \quad (3)$$

where $v-V$ is the final speed in the new frame. In the first term, we have taken into account that initially the speed is $-V$ in the frame considered (whereas it vanishes in a frame fixed to the ground, as applied in Equation (2)).

We would expect to obtain from Equation (3) $v = \sqrt{2gh}$, i.e. the same result as from Eq. (2). However, Equation (3) yields a different result! How can this be solved? I found, to my surprise, that the solution to this paradox also has an application that is usually considered impossible to solve applying the conservation of energy! The solution of the paradox is presented in the next section. The additional application is explained in the last section.

Solving the paradox

The solution to the paradox above is simple (but very few students succeed in solving it by themselves). In the frame moving down with constant speed V , the final height of mass m is not zero, but Vt (because the frame itself has moved a distance Vt downwards during the time interval t of the fall!). Therefore, Equation (3) is not the right form of Equation (1) in this frame. We have to add an additional term,

$$\frac{m(-V)^2}{2} + mgh = \frac{m(v-V)^2}{2} + mgVt, \quad (4)$$

This equation can be written as

$$mgh - mgVt = \frac{mv^2}{2} - mvV. \quad (5)$$

Now the principle of relativity requires that laws of physics are the same in all frames. Thus, this equation must hold for any value of V . Obviously this is possible only if the terms independent of V are the same, and the terms proportional to V are also equal. Thus:

$$mgh = \frac{mv^2}{2} . \quad (5)$$

$$-mgVt = -mvV . \quad (6)$$

The first of these equations yields the right result $v = \sqrt{2gh}$. Therefore, the paradox is solved. No matter in which frame we compute, we obtain the same prediction for the final speed v relative to the ground (in spite of the fact that the kinetic energy depends on the frame considered).

Application

We can go further and use Equation (6) to predict the time it takes the mass m to hit the ground! Indeed, we obtain $t = v/g = \sqrt{2h/g}$. Many textbooks stress that energy conservation is useful to compute speeds or distances, but not time intervals (see e.g. [1]). Resolution of a simple paradox has made it possible to realize that, in fact, energy conservation can be used to predict time intervals also.

[1] Tipler P A 1991 *Physics for scientists and engineers* vol I (New York: Worth) p. 156